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Structure of the string *R*-matrix

Alessandro Torrielli

Center for Theoretical Physics, Laboratory for Nuclear Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
and
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail: torriell@mit.edu

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Abstract

By requiring invariance directly under the Yangian symmetry, we rederive Beisert's quantum R-matrix, in a form that carries explicit dependence on the representation labels, the braiding factors and the spectral parameters u_i. In this way, we demonstrate that there exists rewriting of its entries, such that the dependence on the spectral parameters is purely of a *difference form*. Namely, the latter enter only in the combination $u_1 - u_2$, as indicated by the shift automorphism of the Yangian. When recasted in this fashion, the entries exhibit a cleaner structure, which allows us to spot new interesting relations among them. This permits us to package them into a practical tensorial expression, where the nondiagonal entries are taken care of by explicit combinations of symmetry algebra generators.

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1. Introduction

Integrability in AdS/CFT [1–4] (see [5] for reviews) is strictly related to the existence of an R-matrix [6], which is a solution of the quantum Yang–Baxter equation [7, 8], and satisfies the Hopf-algebraic analog of crossing symmetry [9, 10]. This R-matrix exhibits a certain infinite-dimensional Yangian-type symmetry [11], based on the central extension of the Lie superalgebra $\mathfrak{psu}(2|2)$, plus an additional Yangian generator of $\mathfrak{u}(2|2)$ signature [12, 13]. A universal form of the R-matrix would be desirable, particularly in view of its potential use in studying finite-size effects [14, 15]. Such a universal R-matrix is still unknown, mainly due to the fact that the dual Coxeter number of the algebra vanishes, and therefore traditional mathematical techniques do not apply in a straightforward manner [16–19]. Nevertheless, progress has been made in several aspects [20–24], like in the study of the correspondent classical r-matrix [13, 24–27], in deriving higher representations [28] and in giving the Yangian an almost canonical form [29].

The way the R-matrix was originally derived [6] was by using the Lie superalgebra symmetry in the fundamental representation. The combination of the peculiar features of the tensor products of two fundamentals [7], together with the non-trivial braiding of the coproduct [4, 8, 10], allowed one to fix all entries up to an overall scalar factor [30, 31]. The Yangian symmetry was explicitly discovered only later [11], initially hardly visible due to the relations that mix together all the parameters in the game. In higher representations the Yangian may give useful constraints [27], but the Yang–Baxter equation could probably be sufficient to fix the R-matrix [28].

Were the braiding absent, instead, as in the case of more traditional Yangians, the Lie superalgebra symmetry would then probably not be enough, and the invariance under the Yangian would determine much of the form of the R-matrix. Since the Yangian carries the spectral parameter u, this normally gives the familiar rational R-matrices, depending only on the difference $u_1 - u_2$ due to a constant shift automorphism $u \to u + c$. From there, investigation of the universal R-matrix is easier.

In this paper, we want to exploit the privileged viewpoint that the Yangian gives on the R-matrix, and imagine that we were to fix the latter by purely using the (first-level) Yangian generators. Our strategy is to try not to use any constraint and to carry on the various parameters almost until the end as if they were unrelated. In particular, the relevant Yangian still possesses the shift automorphism [11, 29]. Our calculation produces explicit expressions for the entries, where the dependence on the spectral parameters u_i is indeed purely of a *difference form*, namely $u_1 - u_2$, as expected for the above reasons. Moreover, the other parameters (representation labels, braiding factors) are also recognizable, which helps in finding new relations among the entries and tells us much about the combination of generators that is going to produce them. In this spirit, we also provide tensorial repackaging of the entries, which further clarifies their hidden structure and might serve as a device to investigate the properties of the universal R-matrix.

2. Structural relations

In this section, we will follow the approach of rederiving the R-matrix, not by using the fundamental representation of the Lie superalgebra, as is traditionally done, but rather by using the first Yangian generators from the very beginning. The correspondent coproducts are given in [11, 29]. In doing this, we will leave explicit the dependence on the representation labels, the braiding factors and the spectral parameters, as if they were not related to each other. This will allow us to trace them back through the entries of the R-matrix. Of course, only when they satisfy the familiar relations can one find a solution; therefore, it is impossible to really terminate the process without imposing those relations. Nevertheless, it is possible to proceed far enough to reach a satisfactory form for all the entries, which we report below. At this point, there is no need to continue and we can just read the entries we obtain. As a check, we can then verify that on the constraints we recover Beisert's quantum R-matrix.

Let us first recall the basic definition of the labels of the fundamental representation:

$$\begin{aligned} &\mathfrak{R}^a{}_b|\phi^c\rangle = \delta^c_b|\phi^a\rangle - \tfrac{1}{2}\delta^a_b|\phi^c\rangle, \qquad &&\mathfrak{L}^\alpha{}_\beta|\phi^\gamma\rangle = \delta^\gamma_\beta|\phi^\alpha\rangle - \tfrac{1}{2}\delta^\alpha_\beta|\phi^\gamma\rangle,\\ &\mathfrak{Q}^\alpha{}_a|\phi^b\rangle = a\delta^b_a|\psi^\alpha\rangle, &&\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle = b\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b\rangle, \\ &\mathfrak{S}^a{}_\alpha|\phi^b\rangle = c\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\beta\rangle, &&\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle = d\delta^\beta_\alpha|\phi^a\rangle. \end{aligned} \tag{1}$$

The labels satisfy $ad - bc = 1$. The central (braiding) operator $\mathfrak{U}$ has eigenvalue U, and u will be the spectral parameter.

Let us write the R-matrix in the familiar $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant fashion:

$$\begin{aligned}
\mathcal{R}_{12}\left|\phi_1^a \phi_2^b\right\rangle &= R_{12}^{12}\left|\phi_1^a \phi_2^b\right\rangle + R_{21}^{12}\left|\phi_1^b \phi_2^a\right\rangle + R_{34}^{12}\epsilon^{ab}\epsilon_{\alpha\beta}\left|\psi_1^\alpha \psi_2^\beta\right\rangle,\\
\mathcal{R}_{12}\left|\psi_1^\alpha \psi_2^\beta\right\rangle &= R_{34}^{34}\left|\psi_1^\alpha \psi_2^\beta\right\rangle + R_{43}^{34}\left|\psi_1^\beta \psi_2^\alpha\right\rangle + R_{12}^{34}\epsilon^{\alpha\beta}\epsilon_{ab}\left|\phi_1^a \phi_2^b\right\rangle,\\
\mathcal{R}_{12}\left|\phi_1^a \psi_2^\beta\right\rangle &= R_{13}^{13}\left|\phi_1^a \psi_2^\beta\right\rangle + R_{31}^{13}\psi_1^\beta \phi_2^a\rangle,\\
\mathcal{R}_{12}\left|\psi_1^\alpha \phi_2^b\right\rangle &= R_{31}^{31}\left|\psi_1^\alpha \phi_2^b\right\rangle + R_{13}^{31}\left|\phi_1^b \psi_2^\alpha\right\rangle.
\end{aligned} \tag{2}$$

Choosing the overall normalization to be such that

$$R_{33}^{33} = \frac{U_2}{U_1}, \tag{3}$$

the expressions for the entries which we obtain by imposing invariance under the Yangian are given by the following formulae:

$$\begin{aligned}
R_{31}^{31} &= U_2 \frac{u_1 - u_2}{u_1 - u_2 - a_1 d_1 + a_1 b_1 (c_2/a_2) U_2^2},\\
R_{13}^{31} &= R_{31}^{31} \frac{1}{U_2} \frac{\left(a_2 d_1 - b_1 c_2 U_2^2\right)}{u_1 - u_2},\\
R_{13}^{13} &= \frac{U_2^2}{U_1} \frac{u_1 - u_2}{u_1 - u_2 + b_2 c_2 - a_2 b_2 (d_1/b_1) U_1^2},\\
R_{31}^{13} &= R_{13}^{13} \frac{1}{U_2} \frac{\left(-b_2 c_1 + a_1 d_2 U_2^2\right)}{u_1 - u_2},\\
R_{34}^{12} &= R_{12}^{12} \frac{1}{U_2} \frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1 - u_2 - 1},\\
R_{21}^{12} &= R_{12}^{12} \frac{1}{u_1 - u_2} \left(1 + \frac{1}{U_2^2} \frac{\left(c_1 a_2 - a_1 c_2 U_2^2\right)\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1 - u_2 - 1}\right),\\
R_{12}^{34} &= R_{34}^{34} \frac{1}{U_2} \frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1 - u_2 + 1},\\
R_{43}^{34} &= R_{34}^{34} \frac{1}{u_1 - u_2} \left(-1 + \frac{1}{U_2^2} \frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1 - u_2 + 1}\right),\\
R_{11}^{11} &= R_{12}^{12} + R_{21}^{12} = \frac{U_2^2}{U_1^2} \frac{u_1 - u_2 - b_1 c_1 + a_1 b_1 (d_2/b_2) U_2^2}{u_1 - u_2 + b_2 c_2 - a_2 b_2 (d_1/b_1) U_1^2}.
\end{aligned} \tag{4}$$

As one can immediately see, the dependence of the entries on the spectral parameters u_i is purely of a *difference form*, as indicated by the fact that the Yangian is invariant under an automorphism that shifts the spectral parameter by a constant. Of course, these formulae have to be evaluated on the constraint surface for the fundamental representation, as we will do below. There, the representation labels themselves depend on the spectral parameters, and the difference form is lost. These formulae however single out a part which depends on the difference of suitable spectral variables, as in the case of traditional Yangian R-matrices, and are therefore suggestive of a possible universal origin, the remaining part being ascribed to the particular algebra representation and its braiding factors.

One can check that these expressions indeed reproduce the entries of Beisert's R-matrix [6]. It is sufficient to recall the parameterization

$$a=\sqrt{g}\gamma,\qquad b=\sqrt{g}\frac{\alpha}{\gamma}\left(1-\frac{x^+}{x^-}\right),\qquad c=\sqrt{g}\frac{\mathrm{i}\gamma}{\alpha x^+},\qquad d=\sqrt{g}\frac{x^+}{\mathrm{i}\gamma}\left(1-\frac{x^-}{x^+}\right),\tag{5}$$

together with $U=\sqrt{x^+/x^-}$ and the relation $x^++1/x^+-x^--1/x^-=i/g$, and that $u=(\mathrm{i}g/2)(x^++x^-)(1+1/x^+x^-)$. Then, we compare with the familiar formulae:

$$\begin{aligned}
&\mathcal{R}_{12}\left|\phi_1^a\phi_2^b\right\rangle=\tfrac12(A_{12}-B_{12})\left|\phi_1^a\phi_2^b\right\rangle+\tfrac12(A_{12}+B_{12})\left|\phi_1^b\phi_2^a\right\rangle+\tfrac12C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}\left|\psi_1^\alpha\psi_2^\beta\right\rangle,\\
&\mathcal{R}_{12}\left|\psi_1^\alpha\psi_2^\beta\right\rangle=-\tfrac12(D_{12}-E_{12})\left|\psi_1^\alpha\psi_2^\beta\right\rangle-\tfrac12(D_{12}+E_{12})\left|\psi_1^\beta\psi_2^\alpha\right\rangle-\tfrac12F_{12}\epsilon^{\alpha\beta}\epsilon_{ab}\left|\phi_1^a\phi_2^b\right\rangle,\\
&\mathcal{R}_{12}\left|\phi_1^a\psi_2^\beta\right\rangle=G_{12}\left|\phi_1^a\psi_2^\beta\right\rangle+H_{12}\left|\psi_1^\beta\phi_2^a\right\rangle,\\
&\mathcal{R}_{12}\left|\psi_1^\alpha\phi_2^b\right\rangle=L_{12}\left|\psi_1^\alpha\phi_2^b\right\rangle+K_{12}\left|\phi_1^b\psi_2^\alpha\right\rangle,
\end{aligned}\tag{6}$$

with the functions $A_{12}, B_{12},\ldots$ given by

$$\begin{aligned}
&A_{12}=\frac{x_2^+-x_1^-}{x_2^--x_1^+},\qquad B_{12}=\frac{x_2^+-x_1^-}{x_2^--x_1^+}\left(1-2\frac{1-1/x_1^+x_2^-}{1-1/x_1^+x_2^+}\frac{x_2^--x_1^-}{x_2^+-x_1^-}\right),\\
&C_{12}=\frac{2\gamma_1\gamma_2U_2}{\alpha x_1^+x_2^+}\frac{1}{1-1/x_1^+x_2^+}\frac{x_2^--x_1^-}{x_2^--x_1^+},\\
&D_{12}=-\frac{U_2}{U_1},\qquad E_{12}=-\frac{U_2}{U_1}\left(1-2\frac{1-1/x_1^-x_2^+}{1-1/x_1^-x_2^-}\frac{x_2^+-x_1^+}{x_2^--x_1^+}\right),\\
&F_{12}=-\frac{2\alpha\left(x_1^+-x_1^-\right)\left(x_2^+-x_2^-\right)}{\gamma_1\gamma_2U_1x_1^-x_2^-}\frac{1}{1-1/x_1^-x_2^-}\frac{x_2^+-x_1^+}{x_2^--x_1^+},\\
&G_{12}=\frac{1}{U_1}\frac{x_2^+-x_1^+}{x_2^--x_1^+},\qquad H_{12}=\frac{\gamma_1U_2}{\gamma_2U_1}\frac{x_2^+-x_2^-}{x_2^--x_1^+},\\
&L_{12}=U_2\frac{x_2^--x_1^-}{x_2^--x_1^+},\qquad K_{12}=\frac{\gamma_2}{\gamma_1}\frac{x_1^+-x_1^-}{x_2^--x_1^+}.
\end{aligned}\tag{7}$$

By making use of (4), (5) and (7), it is possible to verify that the two R-matrices indeed exactly match.

As an interesting remark, we note that in formula (4) it is possible to *formally* switch off the braiding ($U_i\to1$), and also send b_i, c_i to zero[1] everywhere and a_i, d_i to 1. This should correspond to scatter two representations of $\mathfrak{gl}(2|2)$. Indeed, if one does that, one finds out that the R-matrix becomes formally equal to

$$\mathcal{R}_{12}=\frac{u_1-u_2}{u_1-u_2-1}\left(1\otimes1+\frac{1}{u_1-u_2}\sum_{i,j=1}^{4}(-)^jE_{ij}\otimes E_{ji}\right),\tag{8}$$

where E_{ij} are the unit matrices with all zeroes but 1 in position (i,j), and bosonic and fermionic indices are altogether numbered from 1 to 4. The combination $\sum_{i,j=1}^4(-)^jE_{ij}\otimes E_{ji}$ is the quadratic Casimir of $\mathfrak{gl}(2|2)$, which already emerged from one-loop gauge theory and from the classical analysis of [25]. The R-matrix which we obtain by this formal procedure is recognized this time as the quantum (Yang-type) R-matrix of $\mathfrak{gl}(2|2)$ in the fundamental representation, and it is an easy exercise to check that it solves the Yang–Baxter equation. We have thus found a consistent practical way of tuning on and off the central extensions (which are proportional to b and c respectively) in the formula for the R-matrix.

[1] Undetermined expressions like b_1/b_2 are sent to 1.

3. Tensorial repackaging

We would like to exploit here the rewriting achieved in the previous section, in order to express the whole R-matrix in a more compact tensorial form. This will be far from enough to be able to provide the universal R-matrix, but will be a useful exercise which may teach us where some of the terms are likely to come from. We (re)write below some of the important identities that one obtains from the expressions given in the previous section:

$$
\begin{aligned}
R^{31}_{13} &= R^{31}_{31}\frac{1}{U_2}\frac{\left(a_2 d_1 - b_1 c_2 U_2^2\right)}{u_1-u_2},\\
R^{13}_{31} &= R^{13}_{13}\frac{1}{U_2}\frac{\left(-b_2 c_1 + a_1 d_2 U_2^2\right)}{u_1-u_2},\\
R^{12}_{34} &= R^{12}_{12}\frac{1}{U_2}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1-u_2}\frac{u_1-u_2}{u_1-u_2-1},\\
R^{34}_{12} &= R^{34}_{34}\frac{1}{U_2}\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1-u_2}\frac{u_1-u_2}{u_1-u_2+1},\\
R^{12}_{21} &= R^{12}_{12}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1-u_2}\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1-u_2}\frac{u_{-}u_2}{u_1-u_2-1} + R^{12}_{12}\frac{1}{u_1-u_2},\\
R^{34}_{43} &= R^{34}_{34}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1-u_2}\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1-u_2}\frac{u_{-}u_2}{u_1-u_2+1} + R^{34}_{34}\frac{1}{u_1-u_2}.
\end{aligned}
\tag{9}
$$

We remark that these formulae have been obtained by imposing invariance only with respect to the Yangian generators, namely requiring $[\Delta(\hat{\mathcal{J}}), \mathcal{R}_{12}] = 0$, where $\hat{\mathcal{J}}$ evaluates to $u\mathcal{J}$ on the fundamental representation and $\mathcal{J}$ being any of the Lie algebra generators. Invariance with respect to the Lie algebra $[\Delta(\mathcal{J}), \mathcal{R}_{12}] = 0$ has not been used. Closure of the algebra and constraints, on the other hand, are needed to fully prove the invariance, as explained at the beginning of section 2. Only when neglecting the constraints would one be able to consider the representation labels and the spectral parameters as independent variables. Therefore, these formulae are valid on the constraint surface for the fundamental representation, but the particular form emerging from this analysis is suggestive of their possible universal origin. In fact, the above pattern is not too unfamiliar, when looking at the standard literature [17–19]. After trying different possibilities, one can see that a relatively simple tensor that produces this relation can be found:

$$
\mathcal{R}_{12} = \mathcal{R}^F_1 \mathcal{R}^F_2 \mathcal{R}^H_{12} + \mathcal{R}^B_1 \mathcal{R}^H_1 + \mathcal{R}^B_2 \mathcal{R}^H_2, \tag{10}
$$

where

$$
\begin{aligned}
\mathcal{R}^F_1 &= 1\otimes 1 + \frac{1}{u_1-u_2}\left(\mathfrak{Q}^1{}_1\otimes\mathfrak{S}^1{}_1\mathfrak{U} + \mathfrak{Q}^2{}_2\otimes\mathfrak{S}^2{}_2\mathfrak{U} - \mathfrak{S}^1{}_1\otimes\mathfrak{Q}^1{}_1\mathfrak{U}^{-1} - \mathfrak{S}^2{}_2\otimes\mathfrak{Q}^2{}_2\mathfrak{U}^{-1}\right),\\
\mathcal{R}^F_2 &= 1\otimes 1 + \frac{1}{u_1-u_2}\left(\mathfrak{Q}^1{}_2\otimes\mathfrak{S}^2{}_1\mathfrak{U} + \mathfrak{Q}^2{}_1\otimes\mathfrak{S}^1{}_2\mathfrak{U} - \mathfrak{S}^1{}_2\otimes\mathfrak{Q}^2{}_1\mathfrak{U}^{-1} - \mathfrak{S}^2{}_1\otimes\mathfrak{Q}^1{}_2\mathfrak{U}^{-1}\right),\\
\mathcal{R}^B_1 &= \frac{\Pi_B\otimes\Pi_B}{1-u_1+u_2} + \frac{1}{u_1-u_2}\left(\mathfrak{R}^1{}_2\otimes\mathfrak{R}^2{}_1 + \mathfrak{R}^2{}_1\otimes\mathfrak{R}^1{}_2\right), \qquad \mathcal{R}^H_1 = R^{12}_{12} 1\otimes 1,\\
\mathcal{R}^B_2 &= \frac{\Pi_F\otimes\Pi_F}{1+u_1-u_2} - \frac{1}{u_1-u_2}\left(\mathfrak{L}^1{}_2\otimes\mathfrak{L}^2{}_1 + \mathfrak{L}^2{}_1\otimes\mathfrak{L}^1{}_2\right), \qquad \mathcal{R}^H_2 = R^{34}_{34} 1\otimes 1,\\
\mathcal{R}^H_{12} &= R^{13}_{13}\Pi_B\otimes\Pi_F + R^{31}_{31}\Pi_F\otimes\Pi_B + \mathcal{R}^H_B,
\end{aligned}
\tag{11}
$$

Π_B and Π_F being the projectors onto the bosonic and fermionic subspaces respectively, $\Pi_B = \text{diag}\{1, 1, 0, 0\}$ and $\Pi_F = \text{diag}\{0, 0, 1, 1\}$. The only non-zero entries of $\mathcal{R}_B^H$ are

$$\begin{aligned}
&\mathcal{R}_B^H \left|\phi_1^a \phi_2^a\right\rangle = \left(R_{11}^{11} + \frac{R_{12}^{12}}{u_1 - u_2 - 1}\right)\left|\phi_1^a \phi_2^a\right\rangle, \\
&\mathcal{R}_B^H \epsilon_{ab}\left|\phi_1^a \phi_2^b\right\rangle = R_{12}^{12}\frac{u_1 - u_2}{u_1 - u_2 - 1}\epsilon_{ab}\left|\phi_1^a \phi_2^b\right\rangle, \\
&\mathcal{R}_B^H \left|\psi_1^\alpha \psi_2^\alpha\right\rangle = \left(R_{33}^{33} - \frac{R_{34}^{34}}{u_1 - u_2 + 1}\right)\left|\psi_1^\alpha \psi_2^\alpha\right\rangle, \\
&\mathcal{R}_B^H \epsilon_{\alpha\beta}\left|\psi_1^\alpha \psi_2^\beta\right\rangle = R_{34}^{34}\frac{u_1 - u_2}{u_1 - u_2 + 1}\epsilon_{\alpha\beta}\left|\psi_1^\alpha \psi_2^\beta\right\rangle.
\end{aligned} \tag{12}$$

This has to be seen as a practical tool to organize the entries, useful for a subsequent attempt to find the universal R-matrix. The Cartan part is traditionally the most complicated to reproduce, as shown by formula (4). But the root part also seems to be little different from standard cases, even when taking into account the fact that in the fundamental representation the roots are nilpotent, and therefore what appears to be linear could well be the first term of a series expansion. Nevertheless, in the classical limit, the formula seems to reproduce the nondiagonal part of the classical r-matrix. In any case, no conclusions are to be drawn at the moment on the persistence of this kind of pattern at the universal level, and the device is purely practical.

4. Conclusions

In this paper we have shown that there exists a way of rewriting Beisert's quantum R-matrix, such that the dependence on the spectral parameters is purely of a difference form. This was hidden before by the complicated relations connecting the spectral parameters with the representation labels and the braiding factors, and can be achieved by using the Yangian symmetry. In particular, this dependence is expected from the shift automorphism that the latter possesses. The remaining dependence on the two representations, which is responsible for the ultimately observed non-difference pattern, is encoded in a precise sequence of representation labels and braiding factors. This, in turn, is to be expected, if the R-matrix has to come from some universal combination of symmetry generators. We provide some hints at this structure, suggested by new relations among the entries which in this new rewriting are easier to spot.

When comparing with the classical r-matrix analysis of [13], one notes that a suitable readjustment of the classical parameters can be used to abandon a pure difference form, in order to reach a convenient normalization for the extra generator to insert in the classical Yangian. The quantum version of this generator [12, 13] has not yet been canonically embedded in the quantum Yangian, and it is hard to judge its role at the moment. In other words, it is possible that one may have to eventually mildly break the pure difference form in order to get a truly universal expression. What we have shown here is that, nevertheless, *there exists* at least one rewriting where the spectral parameters come in differences, all the rest being taken care of by recognizable algebraic quantities.

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Appendix A

In this appendix, we provide evidence that a structure, similar to that we have been discussing for the R-matrix in the fundamental representation, can be identified in the scattering matrix of a fundamental magnon with a bound state of two magnons. This matrix is called S^{AB} in [28]. Let us first write down its action on states (the square brackets are used to represent the bound state of two magnons in the second factor of the tensor product):

$$\begin{aligned}
\mathcal{R}_{12}\big|\phi_1^a[\phi^a\phi^a]_2\big\rangle &= a_1\big|\phi_1^a[\phi^a\phi^a]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\phi^a\phi^b]_2\big\rangle &= \tfrac{1}{3}(2a_1+a_2)\big|\phi_1^a[\phi^a\phi^b]_2\big\rangle\\
&\quad + a_{11}\epsilon^{ab}\big|\phi_1^a[\psi^1\psi^2]_2\big\rangle + a_{13}\epsilon^{ab}\epsilon_{\gamma\delta}\big|\psi_1^\gamma[\phi^a\psi^\delta]_2\big\rangle + \tfrac{1}{3}(a_1-a_2)\big|\phi_1^b[\phi^a\phi^a]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\phi^b\phi^b]_2\big\rangle &= \tfrac{1}{3}(a_1+2a_2)\big|\phi_1^a[\phi^b\phi^b]_2\big\rangle\\
&\quad + 2a_{11}\epsilon^{ab}\big|\phi_1^b[\psi^1\psi^2]_2\big\rangle + 2a_{13}\epsilon^{ab}\epsilon_{\gamma\delta}\big|\psi_1^\gamma[\phi^b\psi^\delta]_2\big\rangle + \tfrac{2}{3}(a_1-a_2)\big|\phi_1^b[\phi^a\phi^b]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\phi^a\psi^\alpha]_2\big\rangle &= a_3\big|\phi_1^a[\phi^a\psi^\alpha]_2\big\rangle + a_{19}\big|\psi_1^\alpha[\phi^a\phi^a]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\phi^b\psi^\alpha]_2\big\rangle &= \tfrac{1}{2}(a_3+a_4)\big|\phi_1^a[\phi^b\psi^\alpha]_2\big\rangle\\
&\quad - a_{16}\epsilon^{ab}\big|\psi_1^\alpha[\psi^1\psi^2]_2\big\rangle + a_{19}\big|\psi_1^\alpha[\phi^a\phi^b]_2\big\rangle + \tfrac{1}{2}(a_3-a_4)\big|\phi_1^b[\phi^a\psi^\alpha]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\phi^a\psi^\alpha]_2\big\rangle &= a_3\big|\phi_1^a[\phi^a\psi^\alpha]_2\big\rangle + a_{19}\big|\psi_1^\alpha[\phi^a\phi^a]_2\big\rangle,\\
\mathcal{R}_{12}\big|\phi_1^a[\psi^1\psi^2]_2\big\rangle &= a_6\big|\phi_1^a[\psi^1\psi^2]_2\big\rangle + a_{15}\epsilon_{\gamma\delta}\big|\psi_1^\alpha[\phi^a\psi^\beta]_2\big\rangle + a_{10}\epsilon_{cd}\big|\phi_1^c[\phi^a\phi^d]_2\big\rangle,\\
\mathcal{R}_{12}\big|\psi_1^\alpha[\phi^a\phi^a]_2\big\rangle &= a_5\big|\psi_1^\alpha[\phi^a\phi^a]_2\big\rangle + 2a_{18}\big|\phi_1^a[\phi^a\psi^\alpha]_2\big\rangle,\\
\mathcal{R}_{12}\big|\psi_1^\alpha[\phi^a\phi^b]_2\big\rangle &= a_5\big|\psi_1^\alpha[\phi^a\phi^b]_2\big\rangle + 2a_{18}\big|\phi_1^{\{a}[\phi^{b\}}\psi^\alpha]_2\big\rangle,\\
\mathcal{R}_{12}\big|\psi_1^\alpha[\phi^a\psi^\alpha]_2\big\rangle &= a_7\big|\psi_1^\alpha[\phi^a\psi^\alpha]_2\big\rangle,\\
\mathcal{R}_{12}\big|\psi_1^\alpha[\phi^a\psi^\beta]_2\big\rangle &= \tfrac{1}{2}(a_7+a_8)\big|\psi_1^\alpha[\phi^a\psi^\beta]_2\big\rangle\\
&\quad + a_{14}\epsilon^{\alpha\beta}\big|\phi_1^a[\psi^1\psi^2]_2\big\rangle + a_{12}\epsilon^{\alpha\beta}\epsilon_{cd}\big|\phi_1^c[\phi^a\phi^d]_2\big\rangle + \tfrac{1}{2}(a_7-a_8)\big|\psi_1^\beta[\phi^a\psi^\alpha]_2\big\rangle,\\
\mathcal{R}_{12}\big|\psi_1^\alpha[\psi^1\psi^2]_2\big\rangle &= a_9\big|\psi_1^\alpha[\psi^1\psi^2]_2\big\rangle - a_{17}\epsilon_{cd}\big|\phi_1^c[\phi^d\psi^\alpha]_2\big\rangle.
\end{aligned} \tag{A.1}$$

In these formulae, $a \neq b$ and $\alpha \neq \beta$ are not summed over. The coefficients a_i have the same meaning as in appendix 6.1.2 of [28][2].

We will focus once again on the nondiagonal part of the R-matrix, the diagonal one being, as expected, increasingly complicated[3]. We make use of the following bound-state representation labels [28] (M = number of bound-state components):

$$\begin{aligned}
a &= \sqrt{\frac{g}{2M}}, & b &= \sqrt{\frac{g}{2M}}\left(1-\frac{x^+}{x^-}\right),\\
c &= \sqrt{\frac{g}{2M}}\frac{\mathrm{i}}{x^+}, & d &= \sqrt{\frac{g}{2M}}\frac{x^+}{\mathrm{i}}\left(1-\frac{x^-}{x^+}\right),
\end{aligned} \tag{A.2}$$

together with $U = \sqrt{x^+/x^-}$, the relations $x^+ + 1/x^+ - x^- - 1/x^- = 2Mi/g, u = -(\mathrm{i}g/4)(x^+ + x^-)(1 + 1/x^+x^-)$ and the same assignment of η factors as in [28]. Taking

[2] The coefficients we will use are obtained from [28] by interchanging x^+ with x^- and g with $-g$.

[3] One can expect this from standard mathematical treatments; see, for instance, theorem 5.2 in [18].

into account that $M = 1$ in the first factor of the tensor product, while $M = 2$ in the second factor, one can prove that the following relations hold:

$$
\begin{aligned}
\frac{a_1 - a_2}{2a_1 + a_2} &= -\frac{1}{U_2}\left[\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 3/2)}\frac{(a_1 + 2a_2)}{(2a_1 + a_2)}\\
&\quad - \left[\frac{1}{u_1 - u_2 - 1/2}\right],\\
\frac{a_{16}}{a_3 + a_4} &= \frac{\mathrm{i}U_1}{2}\left[\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 1/2)},\\
\frac{a_3 - a_4}{a_3 + a_4} &= -\left[\frac{1}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 1/2)},\\
\frac{a_{18}}{a_5} &= \left[\frac{\left(-d_1 a_2 + b_1 c_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 1/2)},\\
\frac{a_{12}}{a_7 + a_8} &= -\frac{\mathrm{i}}{2U_1 U_2}\left[\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 - 3/2)},\\
\frac{a_{17}}{a_9} &= -\frac{\mathrm{i}}{U_2}\left[\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 - 3/2)},\\
\frac{a_{14}}{a_7 + a_8} &= -\frac{1}{2}\left[\frac{\left(d_1 a_2 - b_1 c_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 1/2)},\\
\frac{a_{15}}{a_3} &= -\frac{U_1}{U_2}\left[\frac{\left(-c_1 b_2 + a_1 d_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\right]\frac{(u_1 - u_2 - 3/2)}{(u_1 - u_2 - 1/2 - K_{12})},\\
\frac{a_{13}}{a_1 + 2a_2} &= \frac{\mathrm{i}U_1}{3}\left[\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 3/2)},\\
\frac{a_7 - a_8}{a_7 + a_8} &= -\frac{2}{U_2}\left[\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{(u_1 - u_2 + 1/2)}\right]\frac{(u_1 - u_2 - 1/2)^2}{(u_1 - u_2 - 3/2)(u_1 - u_2 + 1/2)}\\
&\quad + \left[\frac{1}{u_1 - u_2 + 1/2}\right],\\
\frac{a_{11}}{a_1 + 2a_2} &= \frac{\mathrm{i}U_1}{3}\left[\frac{\left(-d_1 a_2 + b_1 c_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\frac{\left(-c_1 a_2 + a_1 c_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\right]\frac{(u_1 - u_2 - 1/2)^2}{(u_1 - u_2 + 3/2)(u_1 - u_2 + 1/2)},\\
\frac{a_{10}}{a_3} &= a_3\frac{\mathrm{i}}{U_1 U_2^2}\left[\frac{\left(-c_1 b_2 + a_1 d_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\frac{\left(d_1 b_2 - b_1 d_2 U_2^2\right)}{(u_1 - u_2 - 1/2)}\right],\\
\frac{a_{19}}{a_3 + a_4} &= -\frac{1}{2U_2}\left[\frac{\left(-c_1 b_2 + a_1 d_2 U_2^2\right)}{u_1 - u_2 - 1/2}\right]\frac{(u_1 - u_2 - 1/2)}{(u_1 - u_2 + 1/2)},
\end{aligned}
\tag{A.3}
$$

where we have defined $K_{12} = U_2^2 a_1 b_1 \frac{d_2}{b_2} - b_1 c_1$. This combination already appears in the diagonal part of the fundamental R-matrix; see, for instance, the expression for R_{11}^{11} in formula (4). Therefore, it is natural to think about it as originating from the Cartan part.

Several remarks are in order when looking at the expressions we have found. First, we have normalized the nondiagonal entries with respect to the correspondent diagonal ones, as one can see from (A.1). This is in the same spirit of formula (9), and allows us to factorize away much of the complication intrinsic to the Cartan part of the R-matrix. Remnants of this complication nevertheless still appear in the parts of formula (A.3) which are *not* enclosed by square brackets. All parts outside the square brackets can in fact be recognized as arising from Cartan elements, as we have already noted and is apparent from (A.1), and as common in standard Yangian computations (see a clear demonstration of this fact in [19])[4]. The difference form with respect to $u_1 - u_2$ is also evident.

Second, let us look at the terms inside the square brackets. They contain the genuine nondiagonal action of the R-matrix. The similarity of their structure with the analogous results for the fundamental R-matrix is remarkable. Once again, one can recognize a factorized part characterized by fermion bilinears, summed to two boson-bilinear terms, in the spirit of (10). The action of these bilinears has to be obtained by using the appropriate (fundamental and bound-state) representations on the two factors of the tensor products. Such representations can be found in [28]. Their increasing tensorial complication and the more and more complicated form of the Cartan elements prevent us so far from proving a complete formula analog to (11), but the similarity in the structure suggests that this will be possible. When comparing with the result obtained for the fundamental R-matrix in section 3, one can see that with our choice the pole has been shifted to $1/(u_1 - u_2 - 1/2)$. As discussed in footnote 4, this shift has a large degree of ambiguity. One is likely to have to adjust these shifts when the number of components of the bound states increase, and more thorough knowledge of the diagonal part will help fixing this dependence.

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[4] There is still a degree of arbitrariness in shifting typical Cartan terms of the form $(u_1 - u_2 + A)/(u_1 - u_2 + B)$ in and out of the square brackets, which cannot be fixed at the moment.

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